Numerical methods for Partial Differential Equations

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The (elastic and thermal) models studied so far are **stationary**, namely they exhibit the following structure:

$$\begin{cases} -\nabla \cdot (\mu \nabla u) = f & \text{in} \quad \Omega \,, \\ \text{boundary conditions} & \text{on} \quad \partial\Omega \,, \end{cases}$$

with $u = u(\boldsymbol{x})$ defined in $\overline{\Omega}$.

Let us now consider situations where the state of our system varies with time. The corresponding models, termed **evolutionary**, exhibit the following structure:

$$\begin{cases} \mathcal{D}_t u - \nabla \cdot (\mu \nabla u) = f & \text{in} \quad \Omega \,, \quad 0 < t \leq T \,, \\ \text{boundary conditions} & \text{on} \quad \partial\Omega \,, \quad 0 < t \leq T, \\ \text{initial conditions} & \text{in} \quad \Omega \,, \quad t = 0 \,, \end{cases}$$

where $\mathcal{D}_t u$ denotes a partial derivative of u with respect to time. Now, $u = u(\boldsymbol{x}, t)$ is defined in $\overline{\Omega} \times [0, T]$.

Let $u = u(\boldsymbol{x}, t)$ be the (small) vertical displacement of a thin membrane at the point $\boldsymbol{x} = (x, y) \in \Omega$ at time t. Then

$$\frac{\partial u}{\partial t}(\boldsymbol{x}, t) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial t^2}(\boldsymbol{x}, t)$$

represent, respectively, the velocity and the acceleration of the membrane in the vertical direction at the point $\boldsymbol{x} = (x, y) \in \Omega$ at time t.
Next, let $\rho = \rho(\boldsymbol{x})$ be the surface density of mass of the membrane, and let $f = f(\boldsymbol{x}, t)$ be the surface density of external force applied at time t.

Then, *Newton's law* is expressed as

$$f + \nabla \cdot \boldsymbol{\tau} = \rho \frac{\partial^2 u}{\partial t^2} \qquad \text{in} \quad \Omega\,, \ \text{at each time } t\,;$$

substituting the expression of $\boldsymbol{\tau}$ given by Hooke's law in this equation, we obtain

$$\rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot (\mu \nabla u) = f \qquad \text{in} \quad \Omega\,, \ \text{at each time } t\,.$$

Such an equation is known as the **wave equation**.

The equation has to be supplemented by

- one boundary condition at each point $\boldsymbol{x} \in \partial\Omega$ (as for the steady case) at each time t,
- *two* conditions at the initial time $t = 0$, at each point $\boldsymbol{x} \in \Omega$, since the equation contains the *second derivative* of u with respect to time.

The initial conditions define the position and the velocity of the membrane at the initial time:

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}) \qquad \text{in } \Omega \,.$$

Thus, if we enforce, e.g., homogeneous Dirichlet boundary conditions, the displacement u will be uniquely determined, for time instants $t \in [0, T]$ with T given, by solving the *initial- and boundary-value problem*

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} - \nabla \cdot (\mu \nabla u) = f & \text{in } \;\Omega \,, \quad 0 < t \le T \,, \\ u = 0 & \text{on } \;\partial\Omega \,, \quad 0 < t \le T, \\ u = u_0, \quad \dfrac{\partial u}{\partial t} = v_0 & \text{in } \;\Omega \,, \quad t = 0 \,. \end{cases}$$

Let $u = u(\boldsymbol{x}, t)$ be the temperature of a thin metallic plate at the point $\boldsymbol{x} \in \overline{\Omega}$ at time t. Let $\rho = \rho(\boldsymbol{x})$ be the surface density of mass of the body, and let $c = c(\boldsymbol{x})$ denote its specific heat, so that $c\rho$ represents its heat capacity per unit of surface.

The equation of thermal balance is

$$c\,\rho\,\frac{\partial u}{\partial t} = -\nabla \cdot \boldsymbol{\Phi} + \rho\, q\ ,$$

where $\boldsymbol{\Phi}$ represents the heat flux and $q = q(\boldsymbol{x}, t)$ denotes an additional contribution of heat per unit of mass from the exterior (e.g., through thermal radiation).

By applying *Fourier's law*

$$\boldsymbol{\Phi} = -\kappa \nabla u\ ,$$

where $\kappa = \kappa(\boldsymbol{x}) > 0$ is the coefficient of thermal conductivity of the body at the point $\boldsymbol{x}$, we obtain the **heat equation**

$$c\,\rho\,\frac{\partial u}{\partial t} - \nabla \cdot (\kappa \nabla u) = \rho\, q \qquad \text{in} \;\; \Omega\ ,\ \text{at each time } t\ .$$

The equation has to be supplemented by

- one boundary condition at each point $\boldsymbol{x} \in \partial\Omega$ (as for the steady case) at each time t,
- *one* condition at the initial time $t = 0$, at each point $\boldsymbol{x} \in \Omega$, namely the initial temperature
$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \ , \quad \boldsymbol{x} \in \Omega \ ,$$
since the equation contains the *first derivative* of u with respect to time.

Let us divide the equation by c, and let us set $\mu = \kappa/c$ and $f = \rho q/c$.

Thus, if we enforce, e.g., homogeneous Dirichlet boundary conditions, the temperature u will be uniquely determined, for time instants $t \in [0, T]$ with T given, by solving the *initial- and boundary-value problem*

$$\begin{cases} \rho \dfrac{\partial u}{\partial t} - \nabla \cdot (\mu \nabla u) = f & \text{in } \ \Omega \ , \quad 0 < t \leq T \ , \\ u = 0 & \text{on } \ \partial\Omega \ , \ 0 < t \leq T \ , \\ u = u_0 & \text{in } \ \Omega \ , \quad t = 0 \ . \end{cases}$$

Discretization strategies

Since the state of the system depends upon space and time

$$u = u(\boldsymbol{x}, t) \ ,$$

it is natural to discretize the problem *in two successive steps*, namely:

first, semi-discretize with respect to one variable,
then, discretize with respect to the other one.

Method of lines:

- *first, semi-discretization with respect to space* (by finite differences, finite elements, finite volumes, ...), obtaining in this way a sistem of ordinary differentiale equations in time;
- *then, discretization with respect to time*, introducing a sequence of discrete time instants.

Method of transverse lines:

- *first, semi-discretization with respect to time*, obtaining in this way a steady problem at each discrete time instant;
- *then, discretization with respect to space* of each steady problem.

Anticipating what we will see in the coming slides, the first step of the method of lines produces an *initial-value problem* (also termed *Cauchy problem*) of the following type:

- **Thermal model:**
$$\begin{cases} \boldsymbol{B}\boldsymbol{u}' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}\ , & 0 < t \leq T\ , \\ \boldsymbol{u}(0) = \boldsymbol{u}_0\ . \end{cases}$$
- **Elastic model:**
$$\begin{cases} \boldsymbol{B}\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}\ , & 0 < t \leq T\ , \\ \boldsymbol{u}(0) = \boldsymbol{u}_0\ , \quad \boldsymbol{u}'(0) = \boldsymbol{v}_0\ . \end{cases}$$

where $\boldsymbol{u} = \boldsymbol{u}(t)$ is a vector in $\mathbb{R}^N$ for each time t, whereas $\boldsymbol{B}$ and $\boldsymbol{A}$ are non-singular square matrices.
Each suffix $'$ denotes one differentiation with respect to the variable t.

Multiplying each equation by $\boldsymbol{B}^{-1}$ (*in a formal way, never do this in practice!*), we obtain that

$$\boldsymbol{B}\boldsymbol{u}' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \qquad \text{becomes} \qquad \boldsymbol{u}' = -\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}^{-1}\boldsymbol{f}$$

and

$$\boldsymbol{B}\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \qquad \text{becomes} \qquad \boldsymbol{u}'' = -\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}^{-1}\boldsymbol{f} \; .$$

Hence, setting

$$\boldsymbol{F}(\boldsymbol{u}, t) = -\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}^{-1}\boldsymbol{f} \; ,$$

we can express the previous Cauchy problems in *canonical form:*

- **Thermal model:**

$$\begin{cases} \boldsymbol{u}' = \boldsymbol{F}(\boldsymbol{u}, t) \; , & 0 < t \leq T \; , \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \; . \end{cases}$$

- **Elastic model:**

$$\begin{cases} \boldsymbol{u}'' = \boldsymbol{F}(\boldsymbol{u}, t) \; , & 0 < t \leq T \; , \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \; , \quad \boldsymbol{u}'(0) = \boldsymbol{v}_0 \; . \end{cases}$$

The semi-discretization in space of the heat equation by finite differences

Let Ω be the square plate $(0, L)^2$. Let us assume that μ is constant, so that the heat equation becomes

$$\rho \frac{\partial u}{\partial t} - \mu \Delta u = f \qquad \text{in } \ \Omega \,, \quad 0 < t \leq T \,.$$

Let us consider the equally-spaced grid $\mathcal{G}_h$ already introduced in the steady case. Let

$$u_{\ell m} = u_{\ell m}(t) \simeq u(x_\ell, y_m, t)$$

be the approximation of the temperature at the node (x_ℓ, y_m) at time t; besides, let

$$u_{\ell m, 0} = u_0(x_\ell, y_m)$$

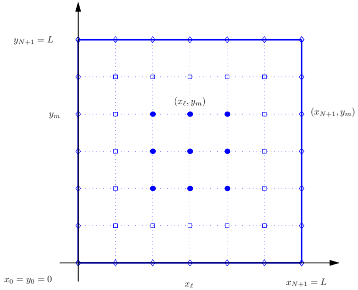
be the initial temperature at the same node, let

$$f_{\ell m} = f_{\ell m}(t) = f(x_\ell, y_m, t)$$

be the right-hand side at time t, and finally let

$$\rho_{\ell m} = \rho(x_\ell, y_m)$$

be the density (constant with respect to time).



At each internal node (x_ℓ, y_m), $1 \le \ell, m \le N$, the discretization in space of the heat equation produces the ordinary differential equation

$$\rho_{\ell,m} u'_{\ell m} + \frac{\mu}{h^2}\left(-u_{\ell,m-1} - u_{\ell-1,m} + 4u_{\ell m} - u_{\ell+1,m} - u_{\ell,m+1}\right) = f_{\ell m} \ , \quad 0 < t \le T \ ,$$

which is supplemented by the initial condition

$$u_{\ell m}(0) = u_{\ell m,0} \ .$$

At the boundary nodes, we enforce, at each time instant $0 < t \le T$, the conditions

$$u_{\ell m}(t) = 0 \qquad \text{if} \quad \ell \in \{0, N+1\} \quad \text{or} \quad m \in \{0, N+1\} \ .$$

As in the steady case, we eliminate the boundary nodes and we collect the internal unknowns, numbered in lexicographical order, in a vector

$$\boldsymbol{u} = \boldsymbol{u}(t) = \big(u_k(t)\big)_{1 \le k \le N^2} \ .$$

In addition, we introduce the vectors

$$\boldsymbol{f} = \boldsymbol{f}(t) = \big(f_j(t)\big)_{1 \le j \le N^2} \qquad \text{and} \qquad \boldsymbol{u}_0 = \big(u_{k,0}\big)_{1 \le k \le N^2} \ .$$

Let us introduce the diagonal matrix which collects the density values at the internal nodes

$$\boldsymbol{B} = \text{diag}\ (\rho_1, \ldots, \rho_j, \ldots, \rho_{N^2})\ ;$$

note that if the density is constant and equal to ρ, we have $\boldsymbol{B} = \rho \boldsymbol{I}$.

Let $\boldsymbol{A}$ denote the penta-diagonal matrix introduced in Chapter 2, whose entries in the upper triangular part are

$$a_{jk} = \frac{\mu}{h^2} \begin{cases} 4 & \text{if } \ k = j\ , \\ -1 & \text{if } \ k = j+1 \text{ with } \ j \neq pN\ , \\ -1 & \text{if } \ k = j+N\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

Then, the previous equations give rise to the system of ordinary differential equations

$$\boldsymbol{B}\boldsymbol{u}' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}\ , \qquad 0 < t \leq T\ ,$$

whereas the initial condition is expressed as

$$\boldsymbol{u}(0) = \boldsymbol{u}_0\ .$$

The semi-discretization in space of the heat equation by finite elements

In a plane polygonal domain, let us consider the initial- and boundary-value problem

$$\begin{cases} \rho \dfrac{\partial u}{\partial t} - \nabla \cdot (\mu \nabla u) = f & \text{in } \Omega\,, \quad 0 < t \le T\,, \\ u = 0 & \text{on } \partial\Omega\,, \; 0 < t \le T\,, \\ u = u_0 & \text{in } \Omega\,, \quad t = 0\,. \end{cases}$$

Notation: if $w(\boldsymbol{x}, t)$ is a function depending upon space and time, for each time instant t let us denote by $w(t)$ the function, defined in $\overline{\Omega}$ such that $(w(t))(\boldsymbol{x}) = w(\boldsymbol{x}, t)$.

The *variational formulation* of the problem is obtained by treating the temporal term in the same way as the forcing term. Precisely, at each time instant $0 < t \le T$, the solution $u(t)$ belongs to the space V of admissible temperatures, and satisfies the equations

$$\int_\Omega \rho \frac{\partial u}{\partial t} v \, d\boldsymbol{x} + \int_\Omega \mu \nabla u \cdot \nabla v \, d\boldsymbol{x} = \int_\Omega f \, v \, d\boldsymbol{x} \qquad \text{for each} \quad v \in V\,.$$

Note that each integral is a function of the time t, through u or f; on the other hand, all test functions v are independent of time.

As in the steady case, the *discrete variational formulation* is obtained from the previous one by substituting V with a finite-dimensional subspace V_h (the space of all *discrete admissible temperatures*).

Such formulation is therefore: for each t, with $0 < t \leq T$,

$$\begin{cases} u_h(t) \in V_h \text{ and satisfies} \\ \displaystyle\int_\Omega \rho \frac{\partial u_h}{\partial t} v_h \, d\boldsymbol{x} + \int_\Omega \mu \nabla u_h \cdot \nabla v_h \, d\boldsymbol{x} = \int_\Omega f \, v_h \, d\boldsymbol{x} \qquad \text{for each } \ v_h \in V_h \ . \end{cases}$$

The condition $u_h(t) \in V_h$ means that the discrete solution u_h is represented as

$$u_h(\boldsymbol{x}, t) = \sum_{k=1}^{N} u_k(t) \varphi_k(\boldsymbol{x}) \ , \tag{59}$$

i.e., it is a linear combination of the basis functions in V_h, whose coefficients are unknown funtions of time. In this manner, one accomplishes a *separation of variables*, between space and time variables.

The partial derivatives of u_h with respect to time give rise to *ordinary derivatives* of its coefficients with respect to such variable. Indeed, one has

$$\frac{\partial u_h}{\partial t}(\boldsymbol{x},t) = \sum_{k=1}^{N} \frac{du_k}{dt}(t)\varphi_k(\boldsymbol{x}) = \sum_{k=1}^{N} u_k'(t)\varphi_k(\boldsymbol{x}) \; .$$

In order to translate the discrete variational formulation into algebraic equations, let us choose as test functions v_h the basis functions φ_j; we obtain, for $0 < t \leq T$,

$$\int_\Omega \rho \, \frac{\partial u_h}{\partial t} \, \varphi_j \, d\boldsymbol{x} + \int_\Omega \mu \nabla u_h \cdot \nabla \varphi_j \, d\boldsymbol{x} = \int_\Omega f \, \varphi_j \, d\boldsymbol{x} \; , \qquad 1 \leq j \leq N \; . \tag{60}$$

Let us observe that

$$\int_\Omega \rho \, \frac{\partial u_h}{\partial t} \, \varphi_j \, d\boldsymbol{x} = \sum_{k=1}^{N} u_k' \int_\Omega \rho \, \varphi_k \varphi_j \, d\boldsymbol{x} = \sum_{k=1}^{N} b_{jk} u_k' \; ,$$

where

$$b_{jk} = \int_\Omega \rho \, \varphi_k \varphi_j \, d\boldsymbol{x}$$

are the entries of a matrix $\boldsymbol{B}$, termed the **mass matrix**.

Then, as in the steady case, one has

$$\int_\Omega \mu \nabla u_h \cdot \nabla \varphi_j \, d\boldsymbol{x} = \sum_{k=1}^{N} u_k \int_\Omega \mu \nabla \varphi_k \cdot \nabla \varphi_j \, d\boldsymbol{x} = \sum_{k=1}^{N} a_{jk} u_k \ ,$$

where a_{jk} are the entries of the already known stiffness matrix $\boldsymbol{A}$. At last, we set

$$f_j(t) = \int_\Omega f(\boldsymbol{x}, t) \varphi_j(\boldsymbol{x}) \, d\boldsymbol{x} \ , \qquad 1 \leq j \leq N \ .$$

Thus, we are led to introduce the vector functions

$$\boldsymbol{u}(t) = (u_k(t))_{1 \leq k \leq N} \qquad \text{and} \qquad \boldsymbol{f}(t) = (f_j(t))_{1 \leq j \leq N} \ .$$

The system of ordinary differential equations (60) is then expressed in the vectorial form

$$\boldsymbol{B}\boldsymbol{u}' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \ , \qquad 0 < t \leq T \ ,$$

already mentioned a few slides above.

As far as the initial condition $u(0) = u_0$ is concerned, eq. (59) for $t = 0$ yields

$$u_h(\boldsymbol{x}, 0) = \sum_{k=1}^{N} u_k(0)\varphi_k(\boldsymbol{x}) \ ,$$

with $u_k(0) = u_h(\boldsymbol{x}_k, 0)$. Then, it is natural to impose that $u_h(0)$ coincides with u_0 at the (internal) nodes of the grid, i.e., that

$$u_h(\boldsymbol{x}_k, 0) = u_0(\boldsymbol{x}_k) \ , \qquad 1 \leq k \leq N \ .$$

Hence, the initial condition of our problem is translated into an initial condition for each of the coefficients $u_k(t)$ which appear in eq. (59), namely

$$u_k(0) = u_0(\boldsymbol{x}_k) = u_{0k} \ , \qquad 1 \leq k \leq N \ ,$$

i.e., in vector notation,

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 = (u_{0k})_{1 \leq k \leq N} \ .$$

The mass matrix $\boldsymbol{B}$ is defined as

$$\boldsymbol{B} = (b_{jk}) \in \mathbb{R}^{N\times N}\,, \qquad \text{with} \;\; b_{jk} = \int_\Omega \rho\, \varphi_k\, \varphi_j\, d\boldsymbol{x}\,.$$

As the stiffness matrix $\boldsymbol{A}$, the mass matrix $\boldsymbol{B}$ is *symmetric, positive definite and sparse*.

The pattern of the matrix (i.e., the positions of the entries which are a-priori different from 0) is the same as for $\boldsymbol{A}$, since we have

$$b_{jk} = \sum_{T\in\mathcal{T}(j)\cap\mathcal{T}(k)} \int_T \rho\, \varphi_k\, \varphi_j\, d\boldsymbol{x}\,.$$

The procedure for building this matrix is also identical to that for the stiffness matrix, being based on the computation of the elemental mass matrices $\boldsymbol{B}^{(T)}$ on the individual elements T of the triangulation, followed by their assemblage.

The elemental mass matrix

Using the same notations of Chapter 2, let

$$b^{(T)}_{\alpha,\beta} = \int_T \rho\, \varphi_\beta\, \varphi_\alpha\, d\boldsymbol{x} \ , \qquad 1 \le \alpha, \beta \le 3 \ ,$$

be the entries of the mass matrix $\boldsymbol{B}^{(T)} \in \mathbb{R}^{3\times 3}$ of the element $T \in \mathcal{T}$.

We may assume that the density ρ is a constant ρ_T in T, since otherwise we approximate it by its mean value on T

$$\rho_T = \frac{1}{\mathsf{area}(T)} \int_T \rho\, d\boldsymbol{x} \ ,$$

or by its value $\rho_T = \rho(\boldsymbol{x}_b)$ in the barycenter of the triangle, or even by the arithmetic mean $\rho_T = \frac{1}{3}\big(\rho(\boldsymbol{x}_1) + \rho(\boldsymbol{x}_2) + \rho(\boldsymbol{x}_3)\big)$ of its values in the vertices of the triangle.

In any case, we set

$$b^{(T)}_{\alpha,\beta} = \rho_T \int_T \varphi_\beta\, \varphi_\alpha\, d\boldsymbol{x} \ .$$

By computing the integrals analytically, one finds

$$b_{\alpha,\beta}^{(T)} = \rho_T \,\mathsf{area}(T) \begin{cases} \frac{1}{6} & \text{if } \alpha = \beta \,, \\ \frac{1}{12} & \text{if } \alpha \neq \beta \,, \end{cases}$$

i.e.,

$$\boldsymbol{B}^{(T)} = \rho_T \,\mathsf{area}(T) \begin{pmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \end{pmatrix} .$$

Hence, the elemental mass matrix depends upon the element only through its area and the mean value of density.

The computation of such a matrix is therefore particularly simple.

In some cases, it is convenient to replace the mass matrix $\boldsymbol{B}$ by a diagonal matrix $\widetilde{\boldsymbol{B}}$, which surrogates its effects. Such a new matrix is obtained by "concentrating the mass" at the nodes, i.e., by replacing each diagonal entry of $\boldsymbol{B}$ by the sum of the entries which sit on the correponding row.

This procedure takes the name of *mass lumping*, and the resulting matrix $\tilde{\boldsymbol{B}}$ is termed *lumped mass matrix*.

The enntries of the lumped mass matrix are therefore given by

$$\tilde{b}_{jj'} = \begin{cases} \displaystyle\sum_{k=1}^{N} b_{jk} = \int_\Omega \rho \left(\sum_{k=1}^{N} \varphi_k \right) \varphi_j \, d\boldsymbol{x}; , & \text{if } j' = j \ , \\ 0 & \text{if } j' \neq j \ . \end{cases}$$

Mass lumping can be accomplished element-by-element, since it is easily seen that *by assembling local diagonal matrices one gets a global diagonal matrix*.

Assume that T is a triangle whose three vertices all carry an unknown (i.e., the three local basis functions are restrictions to T of global basis functions in V_h).
The previously defined matrix $\boldsymbol{B}^{(T)}$ is then replaced by the matrix

$$\widetilde{\boldsymbol{B}}^{(T)} = \rho_T \,\mathsf{area}(T)\, \frac{1}{3} \boldsymbol{I} \tag{61}$$

(where $\boldsymbol{I} \in \mathbb{R}^{3\times 3}$ denotes the identity matrix).

Note that the diagonal entry $\tilde{b}_{\alpha\alpha}^{(T)}$ is given by

$$\tilde{b}_{\alpha\alpha}^{(T)} = \rho_T \sum_{\beta=1}^{3} \int_T \varphi_\beta \, \varphi_\alpha \, d\boldsymbol{x} = \rho_T \int_T \left(\sum_{\beta=1}^{3} \varphi_\beta \right) \varphi_\alpha \, d\boldsymbol{x} = \rho_T \int_T \varphi_\alpha \, d\boldsymbol{x} \; .$$

Recalling that the volume of a pyramid is given by $\frac{1}{3}\mathsf{area}(\text{basis}) \times \text{height}$, one immediately has

$$\int_T \varphi_\alpha \, d\boldsymbol{x} = \frac{1}{3} \mathsf{area}(T) \; ,$$

coherently with eq. (61).

An equivalent manner to get $\widetilde{\boldsymbol{B}}^{(T)}$ from $\boldsymbol{B}^{(T)}$ consists of computing the integrals $\int_T \varphi_\beta \, \varphi_\alpha \, d\boldsymbol{x}$ by numerical quadrature, using the *trapezoidal rule* in two dimensions.

Recalling the values of the basis functions at the vertices of the triangle, one has

$$\int_T \varphi_\beta \, \varphi_\alpha \, d\boldsymbol{x} = \begin{cases} \frac{1}{3}\text{area}(T) \;, & \text{if } \alpha = \beta \;, \\ 0 & \text{if } \alpha \neq \beta \;, \end{cases}$$

which corresponds again to eq. (61).

The mass matrix in dimension 1

If $\Omega = [0, L]$, we already met the mass matrix, when we considered the elastic string problem with restoring term.
Indeed, if we now denote by ρ the restoring coefficient γ, we get

$$\boldsymbol{B} = \{b_{jk}\}_{1 \le j,k \le N} \qquad \text{with} \;\; b_{jk} = \int_0^L \rho \varphi_k \varphi_j \, dx \; .$$

The symmetric positive-definite matrix $\boldsymbol{B}$ turns out to be **tridiagonal**. Precisely, in the case of Dirichlet conditions applied to both endpoints, we have

$$b_{jk} = \begin{cases} \frac{1}{3}(\rho_{j-1/2} h_j + \rho_{j+1/2} h_{j+1}) & \text{if} \;\; k = j \; , \\ \frac{1}{6}\rho_{j-1/2} h_j & \text{if} \;\; k = j-1 \; , \\ \frac{1}{6}\rho_{j+1/2} h_{j+1} & \text{if} \;\; k = j+1 \; , \\ 0 & \text{elsewhere} \; , \end{cases}$$

with

$$\rho_{j-1/2} \sim \frac{1}{h_j} \int_{I_j} \rho(x) \, dx \; , \qquad \rho_{j+1/2} \sim \frac{1}{h_{j+1}} \int_{I_{j+1}} \rho(x) \, dx \; .$$

In the particular case in which ρ is constant in $[0, L]$ and the partition of the interval is equally spaced with step-size h, the previous expression simplifies as

$$b_{jk} = \rho h \begin{cases} \frac{2}{3} & \text{if } \; k = j \; , \\ \frac{1}{6} & \text{if } \; k = j \pm 1 \; , \\ 0 & \text{elsewhere } , \end{cases}$$

i.e., one has

$$\boldsymbol{B} = \rho h \, \text{tridiag} \left[\tfrac{1}{6} \quad \tfrac{2}{3} \quad \tfrac{1}{6} \right] \; .$$

On the other hand, if we apply a Dirichlet condition in the left endpoint and a Neumann condition in the right endpoint, the resulting mass matrix is

$$\boldsymbol{B} = \rho h \, \text{tridiag} \left[\tfrac{1}{6} \quad \tfrac{2}{3} \quad \tfrac{1}{6} \, ; \quad \tfrac{1}{6} \quad \tfrac{1}{3} \right] \; .$$

MODAL ANALYSIS

Let us consider the wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot (\mu \nabla u) = f \qquad \text{in} \;\; \Omega$$

and let us look for the **free periodic motions**, i.e., let us choose f identically zero, and let us look for u as a product of a periodic function depending only on the variable t and a function depending only on the variable $\boldsymbol{x}$, vanishing on the boundary.

Stated equivalently, let us perform a *separation of variables* between time and space, by writing u either in complex form, as

$$u(\boldsymbol{x}, t) = \mathrm{e}^{i\omega t} w(\boldsymbol{x}) \ ,$$

where i denotes the imaginary unit, $\omega/2\pi$ represents a frequency of pulsation and w is a complex-valued function, or in real form, as

$$u(\boldsymbol{x}, t) = \cos(\omega t + \alpha) w(\boldsymbol{x}) \ ,$$

where α represents a phase and w is a real-valued function.

In both cases, u is a periodic function with period $2\pi/\omega$; substituting its expression in the wave equation and simplifying the common (exponential or cosinusoidal) factor depending upon t, we get the equation

$$-\omega^2\rho w - \nabla\cdot(\mu\nabla w) = 0 \qquad \text{in} \quad \Omega$$

satisfied by w.

The boundary condition $u = 0$ yields an analogous condition on w.

In conclusion, setting $\lambda = \omega^2$, the free periodic motions of a membrane fixed along its rim are determined by the (non-trivial) solutions of the eigenvalue problem

$$\begin{cases} -\nabla\cdot(\mu\nabla w) = \lambda\rho w & \text{in} \quad \Omega\,, \\ w = 0 & \text{on} \quad \partial\Omega\,, \end{cases}$$

which are termed the **eigenfunctions** of the elastic membrane.

The study of such a problem gives raise to the *modal* (or *spectral*) *analysis* of the elastic membrane.

Let Ω be a polygonal domain and let us consider the discretization of the previous modal problem by linear finite elements.
At first, let us observe that the variational formulation of the problem is as follows:

$$\begin{cases} w \in V,\ \ \lambda \in \mathbb{R} \text{ and satisfy} \\ \displaystyle\int_\Omega \mu \nabla w \cdot \nabla v \, d\boldsymbol{x} = \lambda \int_\Omega \rho \, w \, v \, d\boldsymbol{x} \qquad \text{for each} \ \ v \in V \ . \end{cases}$$

It is possible to prove that such a problem admits a sequence $\{(\lambda_n, w_n)\}_{n \geq 1}$ of eigensolutions; the eigenvalues are strictly positive and form an increasing sequence which tends to $+\infty$ as $n \to +\infty$, whereas the eigenfunctions are mutually ρ-orthogonal, i.e ., they satisfy

$$\int_\Omega \rho \, w_n \, w_m \, d\boldsymbol{x} = 0 \qquad \text{if } n \neq m \ .$$

The finite element discretization of this problem is as follows:

$$\begin{cases} w_h \in V_h, \ \ \lambda_h \in \mathbb{R} \text{ and satisfy} \\ \displaystyle\int_\Omega \mu \nabla w_h \cdot \nabla v_h \, d\boldsymbol{x} = \lambda_h \int_\Omega \rho \, w_h \, v_h \, d\boldsymbol{x} \qquad \text{for each } \ v_h \in V_h \ . \end{cases}$$

The problem is translated into an algebraic form in the usual way, i.e., by representing w_h in the Lagrange basis as $w_h = \sum_{k=1}^N w_k \varphi_k$ and choosing $v_h = \varphi_j$, $j = 1, \dots, N$, as test functions. One obtains

$$\sum_{k=1}^N w_k \int_\Omega \mu \nabla \varphi_k \cdot \nabla \varphi_j \, d\boldsymbol{x} = \lambda_h \sum_{k=1}^N w_k \int_\Omega \rho \, \varphi_k \varphi_j \, d\boldsymbol{x} \ , \qquad 1 \le j \le N \ .$$

Setting $\boldsymbol{w} = (w_k)_{1 \le k \le N}$, one arrives at the *algebraic generalized eigenvalue problem*

$$\boldsymbol{A}\boldsymbol{w} = \lambda_h \boldsymbol{B}\boldsymbol{w} \ ,$$

where $\boldsymbol{A}$ is the stiffness matrix, whereas $\boldsymbol{B}$ is the mass matrix.

The Rayleigh quotient, introduced above, may be expressed as

$$\lambda_h = \frac{\boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w}}{\boldsymbol{w}^T \boldsymbol{B} \boldsymbol{w}} = \frac{\displaystyle\int_\Omega \mu \|\nabla w_h\|^2 \, d\boldsymbol{x}}{\displaystyle\int_\Omega \rho w_h^2 \, d\boldsymbol{x}} \, .$$

Assuming that all the diameters h_T of the triangles which form the partition of Ω be of the same order of magnitude h, it is possible to show that

$$\lambda_{h,\text{min}} \sim \lambda_{\text{min}} \qquad \text{and} \qquad \lambda_{h,\text{max}} \sim C\, h^{-2} \, ,$$

where λ_{min} denotes the minimal eigenvalue of the exact spectral problem, associated with the minimal frequency of vibration of the membrane.

We will see later on that the second of such relations enforces a severe restriction on the time step in the explicit advancing schemes for the thermal problem.